Some Problems
Related to Geometric Analysis

Jiazu Zhou
Department of Mathematics
Southwest University
(Southwest Normal University)
Chongqing 400715

Visiting scholar of S. S. Chern Institute of Mathematics

College of Mathematics and Statistics
Wuhan University
Wuhan, Hubei 430072

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We are going to present several geometric problems related to integral geometry, geometric convexity, geometric inequalities and geometric analysis (solved and unsolved).

1. Hyperplane Sections of the Cube.

The unit cube $C = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbf{R}^n$ cutting by hyperplane H, which hyperplane cuts C with maximum volume?

for $n = 2$, $\max(Vol_{n-1}(H \cap C)) = \sqrt{2}$ attained by either of the diagonals.

for $n = 3$ the smart reader will guess that the most wonderful section of the cube is the regular hexagon obtained by cutting through the origin (center of the cube C) by a hyperplane H^* orthogonal to a diagonal, that is, $Vol_{n-1}(H^* \cap C) = 3\sqrt{3}/4$. But it is wrong since $\sqrt{2} > 3\sqrt{3}/4$.

It was conjectured for a long time that $\sqrt{2}$ is the optimum for any n.

K. Ball, Cube slicing in R^n, Proc. Amer. Math. Soc., 97(1986), 465-473.

Probability theory and Fourier theory:

$$C \cap H : \sum_{i=1}^{n} a_i x_i = 0 \quad (\sum_{i=1}^{n} a_i^2 = 1),$$

$$Vol(C \cap H) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(a_1 t)}{a_1 t} \cdots \int_{-\infty}^{\infty} \frac{\sin(a_n t)}{a_n t}\, dt,$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^n dt \le \frac{\sqrt{2}}{n} \qquad \text{if} \quad n \ge 2,$$

and equality holds only for $n = 2$.

Though it is apparently obvious it not easy to prove that any hyperplane section through the center of a unit cube has a volume greater than or equal to 1 (it was prove by Vaaler only in 1979: *Vaaler, Jeffrey D. A geometric inequality with applications to linear forms. Pacific J. Math. 83 (1979), no. 2, 543–553*).

$n = 2,\ \ 3$ done by students.

2. Busemann-Petty Problem

If two convex bodies K_i and K_j (say symmetric in the origin) are such that the volume of the section of K_i by any hyperplane H through the origin is always smaller than the volume of the section of K_j by H, that is

$$Vol_{n-1}(K_i \cap H) \leq Vol_{n-1}(K_j \cap H),$$

then the volume of K_i is smaller that the volume of K_j, that is,

$$Vol(K_i) \leq Vol_(K_j).$$

for $n = 2$, it is trivially true (since $K_i \subset K_j$).

for $n \geq 12$, not true (counterexample: D. Larman & C. Rogers, *The existence of a centrally symmetric convex body with central sections that are unexpectedly small, Mathematika, 22(1975), no.2, 164-175*).

for $n \geq 10$, not true (K. Ball (1986)).

for $n \geq 7$, not true (A. Giannopoulos, preprint (1990)).

for $n \geq 5$, not true (J. Bourgain, *On the Busemann-Petty problem for perturbations of the ball, Geom. Funct. Anal. 1(1991), no.1, 1-13*) for $n \geq 4$, not true (Gaoyong Zhang, *Intersection bodies and the Busemann-Petty inequalities in R^4, Ann. of Math. (2) 140 (1994), no. 2, 331–346*).

for $n = 3$, true (R. Gardner, *Intersection bodies and the Busemann-Petty problem, Trans. Amer. Math. Soc. 342 (1994), no. 1, 435–445*).

"The combination of Zhang's work and Gardner's 3-dimensional result is undoubtedly one of the most significant results in this area of geometry" by Paul Goodey.

for $n = 4$, true (by G. Zhang, *A positive solution to the Busemann-Petty problem in R^4. Ann. of Math. (2) 149 (1999), no. 2, 535–543*).

The support function (of a convex set) $K \subset R^n$, is defined as: for $u \in S^{n-1} = \{u = (u_1, \cdots, u_n) : |u| = 1\}$,

$$h_K(u) = h(K, u) = \sup\{(x, u)|\ x \in K\}.$$

A convex set K of R^n is a subset such that $\cup_{x,y \in K} [x, y]$. A convex body is a convex subset with non-empty interior.

3. Hadwiger Containment Problem

Given two (convex) domains $K_i, \quad K_j \subset R^n$, if there is a rigid motion $g \in ISO(R^n)$, such that $gK_j \subset K_i$ or $gK_j \supset K_i$?

Is there a sufficient condition for $gK_j \subset K_i$ or $gK_j \supset K_i$? Hopefully the sufficient condition is a geometric inequality involving the geometric invariants of K_i and K_j (areas, volumes, etc.). If

$$m\{g \in ISO(R^n) : \ gK_j \subset K_i \text{ or } gK_j \supset K_i\} \ > \ 0?$$

for $n = 2$, (Hadwiger 1942)

$$2\pi(A_i + A_j) - L_i L_j \ > \ 0,$$

where A, L are respectively the area, length of K.

(D. Ren, 1987)

$$L_i - L_j \ > \ (\Delta_i + \Delta_j)^{\frac{1}{2}}$$

is a sufficient condition for $gK_j \subset K_i$, where $\Delta = L^2 - 4\pi A$ is the isoperimetric deficit of domain D.

(E. Grinberg, D. Ren & J. Zhou 1998) for domains in a plane X^ϵ of constant curvature ϵ,

$$2\pi(A_i + A_j) - L_i L_j - \epsilon A_i A_j \ > \ 0,$$

is a sufficient condition for $gK_j \subset K_i$ or $gK_j \supset K_i$.

(For $n = 3$) Let K_k ($k = i, j$) be convex domains in $\mathbf{R}^3$ with the C^2 smooth boundaries ∂K_k. Denote by A_k the areas, V_k the volumes, $W_j^{(k)}$ the jth order Minkowski quermassintegrales, and $\tilde{H}_k = \int_{\partial K_k} H_k^2 d\sigma$ the total square mean curvatures of K_k.

(G. Zhang, *A sufficient condition for one convex domain containing another, Chin. Ann. of Math., 4(1988), 447-451*) A sufficient condition for convex body K_i to contain, or to be contained in, convex body K_j is

$$\begin{aligned} 12\pi(V_i + V_j) - 3(W_2^{(i)} A_j + W_2^{(j)} A_i) \\ -(2A_i A_j [3(\tilde{H}_i A_j + \tilde{H}_j A_i) \\ -4\pi(A_i + A_j) - 36 W_2^{(i)} W_2^{(j)}])^{\frac{1}{2}} > 0. \end{aligned}$$

(J. Zhou, *A kinematic formula and analogues of Hadwiger's theorem in space, Cont. Math. Amer. Math. Soc. 140(1992), 159-167*) A sufficient condition for convex body K_i to contain, or to be contained in, convex body K_j is

$$\begin{aligned} 8(V_i + V_j) - 6(W_2^{(i)} A_j + W_2^{(j)} A_i) \\ -\pi(A_i A_j [3(\tilde{H}_i A_j + \tilde{H}_j A_i) - 4\pi(A_i + A_j)])^{\frac{1}{2}} > 0. \end{aligned}$$

(J. Zhou, *When can one domainenclose another in R^3? J. Austral. Math Soc. 59 (1995), 266-272*) A sufficient condition for convex body K_i to contain, or to be contained in, convex body K_j is

$$8(V_i + V_j) - 6(W_2^{(i)} A_j + W_2^{(j)} A_i)$$
$$-\pi r(3(\tilde{H}_i A_j + \tilde{H}_j A_i) - 4\pi(A_i + A_j)) > 0.$$

(J. Zhou, *The Willmore functional and the containment problem in R^4, to appear in* **Science in China**, *2006*) Let K_k $(k = i, j)$ be two convex bodies with the C^2 smooth boundaries ∂K_k in the Euclidean space $\mathbf{R}^4$. Let $\tilde{H}_k$ be the total square mean curvatures of ∂K_k, A_k the areas of ∂K_k, V_k the volumes of K_k, and $W_j^{(k)}$ the jth-order Minkowski quermassintegrales of K_k. Then a sufficient condition for K_i to contain, or to be contained in, K_j is

$$5\pi^2(V_i + V_j) + 14(W_3^{(i)} A_j + W_3^{(j)} A_i) +$$
$$60 W_2^{(i)} W_2^{(j)} - 6(\tilde{H}_i A_j + \tilde{H}_j A_i) > 0.$$

For domains in a 3-space of constant curvature and in R^{2n}, see:

1. J. Zhou, *Sufficient conditions for one domain to contain another in a space of constant curvature, Proc. Amer. Math. Soc., 126 (1998), 2797-2803.*

2. J. Zhou, *Kinematic formulas for mean curvature powers of hypersurfaces and Hadwiger's theorem in* R^{2n}*, Trans. Amer. Math. Soc., 345 (1994), 243-262.*

Open for general cases!

4. The Willmore functional.

Let M be an m-dimensional submanifold, which is assumed to be C^2 smooth, in Euclidean space $\mathbf{R}^n$, and H be the mean curvature of M. If $d\sigma$ denote the volume density of M, we wish to find a lowor bound for the total square mean curvature

$$\int_M H^2 d\sigma.$$

Proposition 1 *(Willmore). Let M be a compact surface in $\mathbf{R}^3$ and H be the mean curvature of M. Then*

$$\int_M H^2 d\sigma \geq 4\pi,$$

with equality if and only if M is a standard sphere.

Willmore initiated the question if an embedded torus in R^3 must have the illmore functional bounded from below by that of the Clifford torus. However, Willmore's original question is still open. The novelty of Willmore question was the fact that the estimate is independent on the metric as well as the embedding. Reference can be easily found in the geometric literature.

If $\dim(M) \neq 2$, then the Willmore functional is not a Riemannian invariant, so by applying homothetic transformation, the value may approach zero. So there is no lower bound in this case. However if we assume that $\mathrm{vol}(M)$, the volume of M, is positive, then the Willmore functional should have a lower bound. The following result is due to B-Y Chen

Proposition 2 *(Chen). Let M be a closed submanifold of dimension m in Euclidean space $\mathbf{R}^n$ and H be the mean curvature of M. Then*

$$\int_M |H|^m d\sigma \geq O_m,$$

with equality if and only if M is imbedded as an m-sphere of $\mathbf{R}^n$.

Here O_m is the area of the m-dimensional unit sphere and its value is

$$O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)},$$

where Γ denote the gamma function.

B-Y Chen also achieved the following inequality:

$$\int_M H^2 d\sigma \geq \frac{\lambda_p}{m} \mathrm{v}ol(M),$$

where p is the lower oder of the immersion (in the case of Chen finite type theory), λ_p is the p-th non-negative eigenvalue of Laplacian and the equality holds when and only when the immersion is of 1-type with order p.

We define the Willmore deficit of a closed surface M of R^3 as

$$WD(M) = \int_M H^2 d\sigma - 4\pi. \qquad (1)$$

Then the result of Willmore will be restated as $WD(M) \geq 0$.

(J. Zhou, *The willmore functional and the containment problem in R^4, to appear in Science in China, 2006*) Let Σ be a convex hypersurface of class C^2, which bounds a convex body K, in the Euclidean space $\mathbf{R}^4$. Let H be the mean curvature of Σ, A the area of Σ, V the volume of K, and W_j the jth-order Minkowski quermassintegrale of K. Then we have

$$\int_\Sigma H^2 d\sigma \geq \frac{7W_3}{3} + \frac{5W_2^2}{A} + \frac{5\pi^2 V}{6A},$$

with equality if Σ is a standard sphere.

J. Zhou, *On Willmore inequality for submanifolds, to appear in Canadian Mathematical Bulletin.*

Theorem 1. Let M be a submanifold of dimension $\frac{n+1}{2}$ in the Euclidean space $\mathbf{R}^n$ and H be the mean curvature of M. Denote by $\tilde{S}$ the total scarlar curvature of M and R the radius of the circumscribed ball of M. Then

$$\int_M H^2 d\sigma \geq \frac{1}{3(n+1)^2}\left(8\tilde{S}(M) + \frac{n+5}{R^2}\mathrm{Vol}(M)\right).$$

Theorem 2. Let M be a submanifold of dimension $\frac{n+1}{2}$ in the Euclidean space $\mathbf{R}^n$ and H be the mean curvature of M. Denote by $\chi(M)$ the Euler characteristic of M and R the radius of the minimum circumscribed ball of M. If $\frac{n+1}{2}$ is even, then we have

$$\begin{aligned}\int_M H^2 d\sigma \geq \frac{1}{3(n+1)^2}\Bigg(&\frac{2^{\frac{n+7}{2}}\pi^{\frac{n+1}{4}}}{(n-1)(n-5)\cdots 2}\chi(M)\\ &+\frac{n+5}{R^2}\mathrm{Vol}(M)\Bigg).\end{aligned}$$

(For a case of R^3 see: J. Zhou, *On the Willmore deficit of convex surfaces, Lect. in Appl. Math. Amer. Math. Soc., 30 (1994), 279-287.*)

5. Minkowski quermassintegrale

A support hyperplane of a convex set K (or a support hyperplane of the convex surface ∂K) is a hyperplane that contains points of K but does not seperate any two points of K. Let K be a convex set and let O be a fixed point in R^n. Consider all the $(n-r)$-plane $L_{n-r[O]}$ through O and let K'_{n-r} be the orthogonal projection of K into $L_{n-r[O]}$. That is, K'_{n-r} denotes the convex set of all intersection points of $L_{n-r[O]}$ with the r-plane perpendicular to $L_{n-r[O]}$ through each point of K. Then the r-th Minkowski quermassintegrale $W_r(K)$, or mean cross-sectional measure, introduced by Minkowski, is defined by the normalized $E(V(K'_{n-r}))$:

$$E(V(K'_{n-r})) = \frac{I_r(K)}{m(G_{n-r,r})},$$

where

$$I_r(K) = \int_{G_{n-r,r}} Vol(K'_{n-r}) dL_{n-r[O]} = \int_{G_{r,n-r}} Vol(K'_{n-r}) dL_{r[O]},$$

$$m(G_{n-r,r}) = \int_{G_{n-r,r}} dL_{n-r[O]}.$$

Alexandrov-Fenchel inequalities:

$$W_i^2 \geq W_{i-1}W_{i+1};\ W_1 = A;\ W_0 = V\ \ (i = 1, \cdots, n-1).$$

open questions?

The Minkowski theory has been an active and fascinating field in Mathematics in the last century and is still going on.

1) R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge University Press, (1993).

2) L. Santaló, Integral Geometry and Geometric Probability, Addison-Wesley Pub. Company (1976).

3) D. Ren, Topic in Integral Geometry, World Scientific, Singapore (1994).

4) D. Klain & G. Rota, Introduction to Geometric Probability, Cambridge University Press, (1997).

5) Yu. D. Burago & V. A. Zalgaller, Geometric Inequalities, Springer-Verlag Berlin Heidelberg (1988).

6. Geometric inequalities

Perhaps the oldest geometric inequality is the following isoperimetric inequality:

Theorem 1 . *The area A and the length L of any domain D in the euclidean plane R^2 satisfy the inequality*

$$L^2 - 4\pi A \geq 0, \tag{2}$$

with the equality if and only if D is a disc.

If we take the convex hull D^* of any domain D, then we have $L^2 - 4\pi A \geq L^{*2} - 4\pi A^*$ since the convex hull increases the area and decreases the length of the boundary. Therefore we only consider the convex domain for the case of isoperimetric inequality.

A more stronger inequality related to isoperimetric inequality is due to Bonnesen:

Theorem 2 . *The area A and the length L of any convex domain D in the euclidean plane R^2 satisfy*

$$L^2 - 4\pi A \geq \pi^2(r_e - r_i)^2, \tag{3}$$

where r_e and r_i are the out-radius and the in-radius of the convex domain D with the quality when and only when $r_e = r_i$, that is, D is a disc.

Another geometric question closely related to the isoperimetric inequality asks: given two domains D_k $(k = i, j)$, when can one domain contain another? More precisely, we ask if there is an isometry g of the plane R^2 so that $gD_j \subset D_i$ or $gD_j \supset D_i$. We wish to have an answer that depends only on the geometric invariants of domains involved, preferably on the areas A_k and the lengths L_k. One would ask the same question for domains in the plane of constant curvature and the domains in R^n. But the invariants may involve more curvature integrals.

Let D_k $(k = i, j)$ be two domains of connected and simply connected and bounded by simple curves. Let G be the group of isometries in R^n and dg be the kinematic measure (Harr measure in measure theory) on G. Then we have the following containment measure

$$\begin{aligned} m\{g \in G \,:\, gD_j \subset D_i \;\text{ or }\; gD_j \supset D_i\} \\ = m\{g \in G \,:\, D_i \cap gD_j \neq \emptyset\} \\ -m\{g \in G \,:\, \partial D_i \cap g\partial D_j \neq \emptyset\}. \end{aligned} \qquad (4)$$

If we can estimate the measure $m\{g \in G \,:\, D_i \cap gD_j \neq \emptyset\}$ from below or (and) the measure $m\{g \in G \,:\, \partial D_i \cap g\partial D_j \neq \emptyset\}$ from above in terms of geometric invariants of D_i and D_j, then we obtain an inequality of the form

$$\begin{aligned} m\{g \in G \,:\, gD_j \subset D_i \;\text{ or }\; gD_j \supset D_i\} \\ \geq f(A_i^1, \cdots, A_i^r; A_j^1, \cdots, A_j^r), \end{aligned} \qquad (5)$$

where each of A_k^α $(k = i, j;\, \alpha = 1, \cdots, r)$ is an integral geometric invariant of D_k.

One can then immediately state the following conclusion:

(I). If $f(A_i^1, \cdots, A_i^r; A_j^1, \cdots, A_j^r) > 0$ then there is an isometry $g \in G$ such that either gD_j contains or is contained in D_i.

(II). If one let $D_i \equiv D_j \equiv D$, then there is no $g \in G$ such that $gD \subset D$ or $gD \supset D$. Therefore we have

$$f(A^1(D), \cdots, A^r(D)) \leq 0. \tag{6}$$

This will result in a geometric inequality.

(III). Let D_i be, respectively, the in-ball and the out-ball of domain D_j ($\equiv D$), i.e., the largest ball contained in D and the smallest ball containing D. Then there is no $g \in G$ such that $gD \subset D_i$ or $gD \supset D_i$ and we have

$$\begin{aligned} f(A^1(D), \cdots, A^r(D), r_e) \leq 0, \\ f(A^1(D), \cdots, A^r(D), r_i) \leq 0. \end{aligned} \tag{7}$$

We would have an Bonnesen-type inequality.

(IV). If we let D_i be a ball of radius r between the inscribed ball of radius r_i and the circumscribed ball of radius r_e of D_j ($\equiv D$), then we have neither $gD \subset D_i$ nor $gD \supset D_i$. Therefore we have an inequality

$$f(A^1(D), \cdots, A^r(D), r) \leq 0; \qquad r_i \leq r \leq r_e.$$

Let $g \in G$, the isometries in R^2. If μ denotes set of all positions of D_j in which either $gD_j \subset D_i$ or $gD_j \supset D_i$ then by the Poincaé formula and the fundamental kinematic formula of Blaschké we have

$$\begin{aligned} m\{g \in G \,:\, gD_j \subset D_i \text{ or } gD_j \supset D_i\} \\ = \int_\mu dg \geq 2\pi(A_i + A_j) - L_i L_j. \end{aligned} \tag{8}$$

This immediately gives

Theorem 3 . *(Hadwiger) Let D_i and D_j be two domains in the euclidean plane R^2. A sufficient condition for D_j to contain, or to be contained in, D_i is*

$$2\pi(A_i + A_j) - L_i L_j > 0. \tag{9}$$

Moreover, if $A_i \geq A_j$, then D_i can contain a copy of D_j.

If we let $D_i \equiv D_j \equiv D$, then D can not contain any copy of D itself. Then the left hand side integral of (7) vanishes and we have Theorem 1, i.e., the isoperimetric inequality (1).

If we let D_i be, respectively, the inscribed disc of radius r_i of D_j ($\equiv D$) and the circumscribed disc of radius r_e of D_j ($\equiv D$), then we have

$$\pi r_i^2 - Lr_i + A \leq 0; \quad \pi r_e^2 - Lr_e + A \leq 0. \tag{10}$$

From these inequality and another general inequality $x^2 + y^2 \geq (x + y)^2/2$, we obtain the Bonnesen's isoperimetric inequality (Theorem 2):

$$L^2 - 4\pi A \geq \pi^2(r_e - r_i)^2. \tag{11}$$

If we let $D_j \equiv D$ and D_i be a disc of radius r satisfying the condition

$$r_i \leq r \leq r_e, \tag{12}$$

then formula (7) immediately gives the following Bonnesen's inequality

$$\pi r^2 - Lr + A \leq 0. \tag{13}$$

The inequality (12) can be rewritten in several equivalent forms:

Theorem 4 . *Let D be a plane domain of area A and bounded by a simple closed curve of length L. Let r_i and r_e are, respectively, the in-radius and out-radius of D. Then for any disc of radius r, $r_i \leq r \leq r_e$, we have the following inequalities*

$$\begin{aligned} Lr &\geq A + \pi r^2; \\ L^2 - 4\pi A &\geq (L - 2\pi r)^2; \\ L^2 - 4\pi A &\geq \left(L - \tfrac{2A}{r}\right)^2; \\ L^2 - 4\pi A &\geq \left(\tfrac{A}{r} - \pi r\right)^2. \end{aligned} \tag{14}$$

Notice that the third inequality of (14) implies

$$\sqrt{L^2 - 4\pi A} \geq \frac{2A}{r_i} - L,$$

$$\sqrt{L^2 - 4\pi A} \geq L - \frac{2A}{r_e}. \tag{15}$$

Adding inequality (15) yields

$$L^2 - 4\pi A \geq A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2. \tag{16}$$

Adding after multiplied by r_i and r_e gives

$$L^2 - 4\pi A \geq L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2. \tag{17}$$

Therefore we have

Theorem 5 . *Let D be a plane domain of area A and bounded by a simple closed curve of length L. Let r_i and r_e are, respectively, the in-radius and out-radius of D. Then we have the following inequalities*

$$L^2 - 4\pi A \geq A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2;$$
$$L^2 - 4\pi A \geq L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2. \tag{18}$$

These geometric inequalities has been generalized to the plane of constant curvature by J. Zhou:

1) J. Zhou, *The Bonnesen-type isoperimetric inequalities, The 10th International Workshop on Differential Geometry, Korea (2006).*

2) J. Zhou, *The Bonnesen-type inequalities*, preprint.

3) J. Zhou & F. Chen, *The Bonnesen-type inequalities in a plane of constant curvature*, preprint.

7. Kinematic formulas

Let M^p, N^q be compact submanifolds of dimensions p, q, respectively, in a homogeneous space G/H and let $I(M^p \cap gN^q)$ be an integral invariant (e.g., volume, surface area, etc.) of the submanifold $M^p \cap gN^q$. Then a lot of works in integral geometry have been concerned with computing integrals of the following type

$$\int_G I(M^p \cap gN^q)\, dg, \tag{19}$$

where dg is the normalized kinematic density of G. For example in the case that G is the group of motions in an n-dimensional Euclidean space R^n, M^p, N^q are submanifolds of R^n and

$$I(M^p \cap gN^q) = \mathrm{Vol}(M^p \cap gN^q)$$

evaluation of (19) leads to the formulas of Poincaré, Blaschké, Santaló and others. Howard obtained a kinematic formula for $I(M^p \cap gN^q) = \mathrm{Vol}(M^p \cap gN^q)$ in a homogeneous space. If $I(M \cap gN) = \chi(M \cap gN)$, the Euler-Poincalé's characteristic of the intersection $M \cap N$ of domains M, N in R^n with smooth boundaries, then (19) leads to S. S. Chern's kinematic fundamental formula. Assume that $I(M^p \cap gN^q) = \mu(M^p \cap gN^q)$ is one of the integral invariant from the Weyl tube formula then

(19) leads to the Chern-Federer kinematic formula for submanifolds in R^n. This integral also leads to the C-S. Chen kinematic formula if we take $I(M \cap gN) = \int_{M \cap gN} \kappa^2 d\sigma$, the total square of the curvature of the intersection curve $M \cap gN$ of two compact surfaces M, N in R^3. T. Shifrin also obtained a kinematic formula by letting $I(M^p \cap gN^q)$ be the integral of a Chern class. Howard and Zhou achieved more general kinematic formula in the case that $I(M^p \cap gN^q)$ is a invariant homogeneous polynomial of the second fundamental forms of $M^p \cap gN^q$ in a homogeneous space.

S. S. Chern, *Kinematic formulas in integral geometry, J. Math. and Mech. 16 (1966), 101-118.* (translated by Zhou in *Mathematics Translations of Chinese Math. Soc.*)

R. Howard, *The kinematic formula in Riemannian homogeneous space, Memoirs Amer. Math. Soc. 509 (1993).*

J. Zhou, *Kinematic formulas for mean curvature powers of hypersurfaces and Hadwiger's theorem in R^{2n}, Trans. Amer. Math. Soc., 345 (1994), 243-262.*

8. Integrals for power of the chords of a convex set.

Let K be a convex set and let σ denote the length of the chord determined by the line G on K. Consider the integrals

$$I_n = \int_{G\cap K\neq\emptyset} \sigma^n dG, \quad J_n = \int_{P_1,P_2\in K} r^n dP_1 \wedge dP_2$$

where n is an integer and r means the distance between points P_1 and P_2 of K.

for the case of $K \subset R^2$,

$$I_0^2 - 4I_1 \geq 0; \quad \text{(isoperimetric inequality)}$$

$$I_n \geq \frac{2\cdot 4\cdots n}{3\cdot 5\cdots(n+1)} 2^{n+1}\pi^{-n} I_1^{(n+1)/2} \qquad \text{for } n = 2, 4, 6, \cdots;$$

$$I_n \geq \frac{1\cdot 3\cdots n}{2\cdot 4\cdots(n+1)} 2^{n}\pi^{-(n+1)} I_1^{(n+1)/2} \qquad \text{for } n = 3, 5, 7, \cdots;$$

$$2^8 I_1^3 \geq 3^2\pi^4 I_2^2,$$

Inequalities among these I_n are works of Santaló (for R^2), D. Wu (for R^3), D. Ren (for R^n) and others.

Open questions?

L. Santaló, Integral Geometry and Geometric Probability, Addison-Wesley Pub. Company (1976).

D. Ren, Topic in Integral Geometry, World Scientific, Singapore (1994).

9. Geometric Measure.

A measure is characterized by two axioms:

$$\text{(A1)}: \qquad \mu(\emptyset) = 0, \quad \emptyset \text{ is the emptyset.}$$

$$\text{(A2)}: \qquad \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

And therefore

$$\mu(\cup_{i=1}^{n} A_i) = \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + \sum_{i<j<k} \mu(A_i \cap A_j \cap A_k) + \cdots$$

for all positive integer.

For a measure to be the volume $\mu_n(A)$ of a solid A in the Euclidean space R^n we must add additional axioms to the definition of a measure.

$$\text{(A3)}: \qquad \mu(A) = \mu(gA), \quad \text{for } g \in G = Iso(R^n).$$

(The volume of a set A is independent of the position of A in R^n.)

$$\text{(A4)}: \qquad \mu_n(P) = x_1 x_2 \cdots x_n,$$

for a parallelotope P with orthogonal sides of length $x_1,\ x_2,\ \cdots,\ x_n$.

By these 4 axioms, we have the following formulas of a sphere S_k of radius (in n-dimensional space R^n):

$$\mu_n(S_k) = \frac{\pi^{n/2} r^{n/2}}{(n/2)!}, \quad n \text{ is even;}$$

$$\mu_n(S_k) = \frac{2^n \pi^{(n-1)/2}((n-1)/2)! r^n}{n!}, \quad n \text{ is odd.}$$

If we keep the first 3 axioms but tamper the forth axiom, the normalization axiom, what will happen?

The elementary symmetric functions of (the following polynomials in) n variables:

$$e_1(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n;$$

$$e_2(x_1, x_2, \cdots, x_n) = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n;$$

$$\cdots$$

$$e_{n-1}(x_1, x_2, \cdots, x_n) = x_2 x_3 \cdots x_n + \cdots + x_1 x_2 \cdots x_{n-1};$$

$$e_n(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n.$$

Letting $\mu_k = e_k(x_1, x_2, \cdots, x_n)$ $(k = 1, 2, \cdots, n)$ will lead to n different invariant measure. Each of the n elementary symmetric functions of n variables leads to the definition of a new invariant measure which is a different generalization of the volume. These n measures are called the intrinsic measures.

The intrinsic measures are independent of each other, except for certain inequalities they satisfy. These inequalities generalized the classical isoperimetric inequality that relates area to volume. Mathematicians are presently studying these as yet unknown inequalities among the intrinsic volumes (geometric inequalities). We know very little about the intrinsic volumes and inequalities among them.

If we add another invariant measure:

$$\mu_0(C) = 1, \qquad C \text{ is non-empty convex set,}$$

and $\mu(\emptyset) = 0$, then we have the Main Theorem of Geometric Probability:

The $n+1$ intrinsic volume $\mu_0, \mu_1, \cdots, \mu_n$ are a basis of the space of all continuous invariant measures defined on all finite unions of compact sets (Hadwiger, D. Klain).

10. Geometric Probability

Consider two compact convex sets A and B in R^n. Let B to be fixed and we randomly drop the rigid set A in the space R^n. What is the probability that A meets B. In other words, we keep B fixed and let A moving under the rigid motion $g \in Iso(R^n)$, find the invariant measure of those g such that $B \cap gA \neq \emptyset$. That is, to find the invariant measure $m\{g \in Iso(R^n) | \ B \cap gA \neq \emptyset\}$. By Hadwiger's theorem of geometric probability, such an invariant measure equals a linear combination of the $n+1$ intrinsic volumes with coefficients independent of B. We determine these coefficients by taking suitable B's (for example, let B be unit balls). This invariant measure is known as a kinematic formula.

One would like to identify these $n+1$ intrinsic volume with Minkowski quermassintegrales.

Thank you for your attention!